ON SOME DYNAMICAL PROPERTIES OF THE DISCONTINUOUS DYNAMICAL SYSTEM REPRESENTS THE LOGISTIC EQUATION WITH DIFFERENT DELAYS

By

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ABSTRACT

In this work the authors are concerned with the discontinuous dynamical system representing the problem of the logistic retarded functional equation $x(t) = \rho x(t-r)\,[1-(t\text{-}2r)],\quad t \in (0,T]$ and $\rho, r > 0,\ \ x(t) = x_0,\ t \le 0.$ *The existence of a unique solution* $x \in L^1[0,\mathrm{T}]$ *which is continuously dependence on the initial data will be proved. The local stability at the equilibrium points will be studied. The bifurcation analysis and chaos will be discussed.*

Keywords: Logistic Functional Equation, Existence, Uniqueness, Equilibrium Points, Local Stability, Bifurcation Analysis.

1. INTRODUCTION

Let $r > 0$ and consider the problem of retarded functional equation

$$x(t) = f(x(t-r)), \qquad t \in (0,T], \tag{1.1}$$

$$x(t) = x_0, \qquad t \le 0. \tag{1.2}$$

Let $t \in (0,r]$, then $t - r \in (-r,0]$ and the solution of (1.1) - (1.2) is given by

$$x(t) = x_r(t) = f(x_0), \qquad t \in (0,r].$$

For $t \in (r,2r]$, we find that $t - r \in (0,r]$ and the solution of (1.1) - (1.2) is given by

$$x(t) = x_{2r}(t) = f(x_r(t)) = f^2(x_0), \quad t \in (r,2r].$$

Repeating the process we can deduce that the solution of the problem (1.1) - (1.2) is given by

$$x(t) = x_{nr}(t) = f^n(x_0), \qquad t \in ((n-1)r, nr],$$

which is continuous on each subinterval $((k-1)r, kr),\ k = 1,2,\ldots,n$, but

$$\lim_{t \to kr^+} x_{(k+1)r}(t) = f^{k+1}(x_0) \ne x_{kr}(t),$$

which implies that the solution of the problem (1.1) - (1.2) is discontinuous (sectionally continuous) on $(0,T]$.

So, we can give the following definition,

Definition 1.1 Let $r_1, r_2, \ldots r_n > 0.$ The discontinuous dynamical system is the problem of retarded functional equation.

$$x(t) = f(t, x(t-r_1), x(t-r_2), \ldots, x(t-r_n)), \quad t \in (0,T] \text{ and } x(t) = x_0, \quad t \le 0 \tag{1.3}$$

Definition 1.2 The equilibrium points of the discontinuous dynamical system (1.3) is the solutions of the equation,

$$x(t) = f(t, x, x, \ldots, x).$$

Consider now the discontinuous dynamical system of the Logistic retarded functional equation with two different delays,

$$x(t) = \rho x(t-r)[1 - x(t-2r)], \quad t \in (0,T] \tag{1.4}$$

$$x(t) = x_0, \qquad t \le 0. \tag{1.5}$$

We study here the existence of a unique continuously dependent solution $x \in L^1$ of the problem (1.4) - (1.5). The asymptotic stability (see [1][2][3][4][5][6][7]) at the equilibrium points will be studied. The bifurcation and chaos, for some different values of r , will be also studied. To compare our results we take r = 1 and we compare the results with the results of the discrete dynamical system of the Logistic difference equation

$$x_n = \rho x_{n-1}(1 - x_{n-2}), \quad n = 1, 2, \ldots . \tag{1.6}$$

2. Existence and Uniqueness

Let $L^1 = L^1[0,T],\ T < \infty$ be the class of Lebesgue integrable functions on [0, T] with norm

$$\|f\| = \int_0^T |f(t)|dt, \quad f \in L^1.$$

Let $D = \{x \in R : 0 \le x(t) \le 1,\ t \in (0,T] \text{ and } x(t) = x_0,\ t \le 0\}.$

Definition 2.1 By a solution of the problem (1.4) - (1.5) we mean a function $x \in L^1$ satisfying the problem (1.4) - (1.5).

Theorem 2.1 The problem (1.4) - (1.5) has a unique solution $x \in L^1$

Proof. Define, on D, the operator $F : L^1 \to L^1$ by

$$Fx(t) = \rho x(t - r)[1 - x(t - 2r)].$$

The operator F makes sense, indeed for $x \in D$ we have

$$|Fx(t)| \le \rho\, |x(t - r)|$$

and

$$\|Fx(t)\| \le \rho(r x_0 + \|x\|).$$

Now for $x, y \in D$, we can obtain

$$\begin{aligned} |Fx - Fy| &\le |\rho x(t - r)(1 - x(t - 2r)) - \rho y(t - r)(1 - y(t - 2r))| \le \\ &\le \rho\, |x(t - r) - y(t - r)| + \rho\, |x(t - 2r) - y(t - 2r)|, \end{aligned}$$

which implies that

$$\begin{aligned} \|Fx - Fy\| &\le \rho \int_0^T |x(t - r) - y(t - r)|dt + \rho \int_0^T |x(t - 2r) - y(t - 2r)|dt = \\ &= \rho[\int_0^r |x(t - r) - y(t - r)|dt + \int_r^T |x(t - r) - y(t - r)|dt + \\ &+ \int_0^{2r} |x(t - 2r) - y(t - 2r)|dt + \int_{2r}^T |x(t - 2r) - y(t - 2r)|dt\,] = \\ &= \rho[\int_r^T |x(t - r) - y(t - r)|dt + \int_{2r}^T |x(t - 2r) - y(t - 2r)|dt\,] \le \\ &\le \rho[\int_0^{T-r} |x(\theta) - y(\theta)|d\theta + \int_0^{T-2r} |x(\varphi) - y(\varphi)|d\varphi\,] \le \\ &\le \rho[\int_0^T |x(\theta) - y(\theta)|d\theta + \int_0^T |x(\varphi) - y(\varphi)|d\varphi\,] \le \\ &\le 2\rho\, \|x - y\|. \end{aligned}$$

If $\rho < \frac{1}{2}$, we deduce that

$$\|Fx - Fy\| \le \|x - y\|$$

and then the problem (1.4) - (1.5) has, on D, a unique solution $x \in L^1$

3. Continuous dependence on initial conditions

Consider the problem

$$\begin{aligned} x(t) &= \rho x(t - r)[1 - x(t - 2r)], \quad t \in (0,T\,], \\ x(t) &= x_0^*, \qquad\qquad t \le 0, \end{aligned} \tag{3.1}$$

For the continuous dependence of The solution of (1.4)-(1.5) on the initial data we have the following theorem.

Theorem 2.2 The solution of the discontinuous dynamical system represents the problem of the logistic retarded functional equation with two different delays is continuously dependent on the initial data.

Proof. Let $x(t)$ and $x^*(t)$ be the solution of the two problems (1.4) -(1.5) and (1.4)-(3.1) respectively, then,

$$\left|x(t)-x^*(t)\right| \le \rho \left|x(t-r)-x^*(t-r)\right| + \rho \left|x(t-2r)-x^*(t-2r)\right|,$$

which implies that

$$\begin{aligned}\left\|x(t)-x^*(t)\right\| &\le \rho\int_0^T \left|x(t-r)-x^*(t-r)\right|dt + \rho\int_0^T \left|x(t-2r)-x^*(t-2r)\right|dt = \\ &= \rho[\int_0^r \left|x(t-r)-x^*(t-r)\right|dt + \int_r^T \left|x(t-r)-x^*(t-r)\right|dt + \\ &\quad + \int_0^{2r} \left|x(t-2r)-x^*(t-2r)\right|dt + \int_{2r}^T \left|x(t-2r)-x^*(t-2r)\right|dt] = \\ &= \rho[\left|x_0-x_0^*\right|\int_0^r dt + \left\|x-x^*\right\| + \left|x_0-x_0^*\right|\int_0^{2r} dt + \left\|x-x^*\right\|] \le \\ &\le \rho(3r)\left|x_0-x_0^*\right| + 2\rho\left\|x-x^*\right\|,\end{aligned}$$

which implies

$$\left\|x-x^*\right\| \le \frac{3\rho r}{1-2\rho}\left|x_0-x_0^*\right|,$$

and prove that

$$\left|x_0-x_0^*\right| \le \delta \quad \Rightarrow \quad \left\|x-x^*\right\| \le \varepsilon = \frac{3\rho r}{1-2\rho}\delta,$$

and the theorem is proved.

4. Equilibrium Points and their asymptotic stability

The equilibrium points of (1.4) are the solution of the equation

$$\rho x_{eq}(1-x_{eq}) = x_{eq},$$

which are

$$(x_{eq})_1 = 0,$$

$$(x_{eq})_2 = 1-\frac{1}{\rho}.$$

The equilibrium point of (1.4) is locally asymptotically stable if all the roots λ of the equation,

$$1 = \rho[(1-x_{eq})\lambda^{-r} - x_{eq}\,\lambda^{-2r}], \qquad (4.1)$$

satisfy $|\lambda| < 1$ (see [8]).

Then the equilibrium point $x_{eq} = 0$ is locally asymptotically stable if $\rho < 1$ while the second equilibrium point $x_{eq} = 1-\frac{1}{\rho}$ is locally asymptotically stable if all the roots λ of the equation,

$$\lambda^{2r} - \lambda^{r} + (\rho-1) = 0. \qquad (4.2)$$

satisfy $|\lambda| < 1$

The equilibrium point $x_{eq} = 0$ is locally asymptotically stable if $0 < \rho < 1$ which is the same as in the discrete case (1.6). Also, when r = 1, we deduce that the equilibrium point $x_{eq} = 1-\frac{1}{\rho},\ \rho > 1$ is locally asymptotically stable if $1 < \rho < 2,$ which is the same as in the discrete case (1.6).

In studying (1.4)- (1.5) it may be useful to study the difference equation (1.6).

5. Bifurcation and Chaos

In this section, some numerical simulations results are presented to show that dynamics behaviors of the discontinuous dynamical system (1.4) -(1.5) change for different values of r and T. To do this, we will use the bifurcation diagrams as follow:-

Example 1

1. we take $r = 1$ and $t \in [0, 200]$, in this case, we get the same behavior as in the discrete case (Figure 1).
2. we take $r = 2$ and $t \in [0, 200]$ (Figure 2).
3. we take $r = 1.5$ and $t \in [0, 200]$ (Figure 3).

Example 2

1. we take $r = 0.1$ and $t \in [0, 20]$ (Figure 4).
2. we take $r = 0.2$ and $t \in [0, 20]$ (Figure 5).
3. we take $r = 0.3$ and $t \in [0,20]$ (Figure 6).

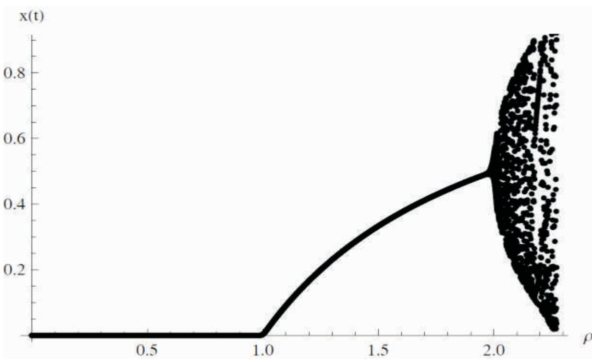

Figure 1. Bifurcation diagram of map (1.4) - (1. 5) with respect to ρ, $r = 1$ and $t \in [0, 200]$

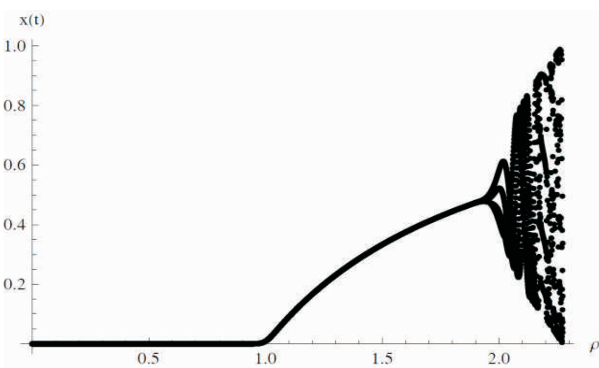

Figure 2. Bifurcation diagram of map (1.4) - (1. 5) with respect to ρ, $r = 2$ and $t \in [0, 200]$

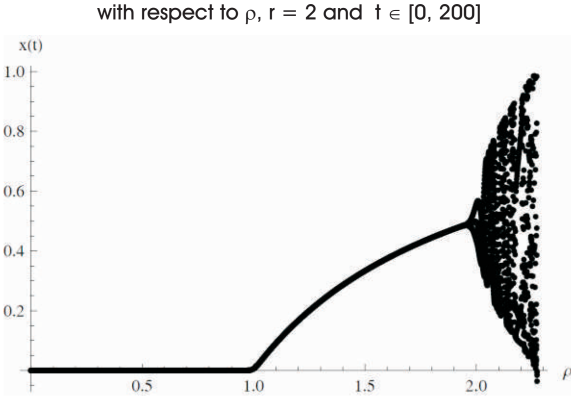

Figure 3. Bifurcation diagram of map (1.4) - (1. 5) with respect to ρ, $r = 1.25$ and $t \in [0, 200]$.

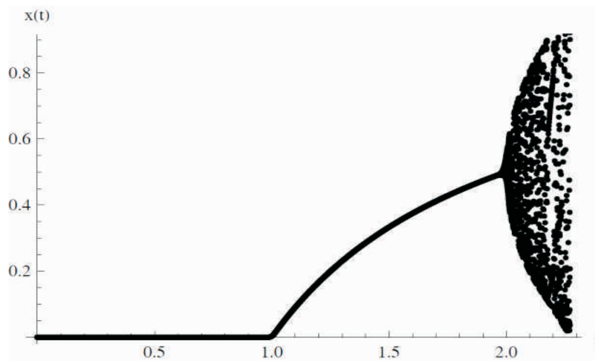

Figure 4. Bifurcation diagram of map (1.4) - (1. 5) with respect to ρ, $r = 0.1$ and $t \in [0, 20]$.

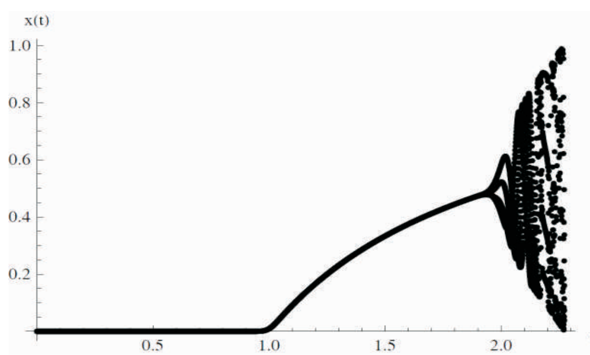

Figure 5. Bifurcation diagram of map (1.4) - (1. 5) with respect to ρ, $r = 0.2$ and $t \in [0, 20]$

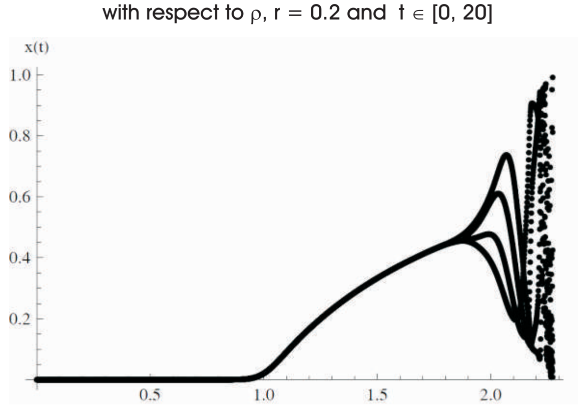

Figure 6. Bifurcation diagram of map (1.4) - (1. 5) with respect to ρ, $r = 0.5$ and $t \in [0, 20]$

From Figures (1 to 6) the authors deduce that the change of r and T effect of stability of the Logistic equation model, occurs of a bifurcation point, parameter sets for which a periodic behavior occur and parameter sets for which a chaotic behavior occur.

Conclusions

The discrete dynamical system of the Logistic equation model describes the dynamical properties for the case $r = 1$ and the time is discrete $t = 1,2,3,4,\ldots$.

On the other hand, the discontinuous dynamical system of the Logistic equation model describes the dynamical properties for different values of the delayed parameter $r \in R^+$ and the time $t \in [0, T]$ is continuous.

Figures (1 & 4) agrees with the results of the asymptotic stability, this confirm that our numerics are correct. Also from figures (1,4 and 2,5), it locks like that there is a scale that gives identical chaos behavior.

This shows the richness of the models of discontinuous dynamical systems.

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